Modern Logistics & Supply Chain Management

ML & SCM

Shortest Route and
Maximal Flows

Dr. Wolfgang Garn
Winter 2016

As gold which he cannot spend
will make no man rich,
so knowledge which he cannot apply
will make no man wise.
Samuel Johnson: The Idler No. 84

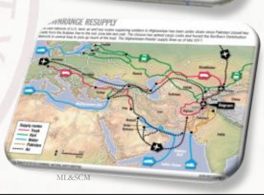
Shortest path problems
Lie at the heart of network flows.
Ahuja, Magnanti, Orlin – Network Flows

Learning Objectives

- To be familiar with a selected number of classical and state-of-the-art methods
- To create solution models
- To develop procedures that offer competitive advantage to the business/organisation

Winter 2014 ML&SCM 2

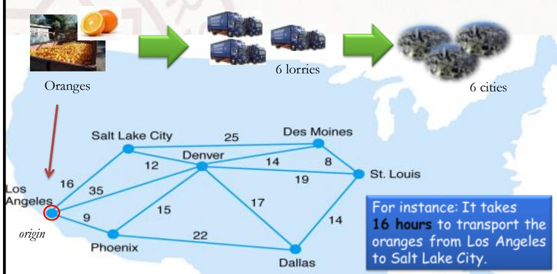
Contents



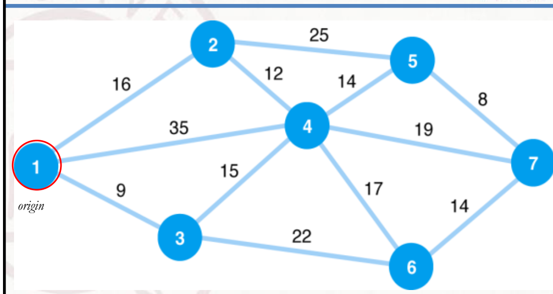
Winter 2016 ML&SCM 3

The Shortest Route Problem
Definition and Example Problem Data (1 of 2)

Problem: Determine the shortest routes from the origin to all destinations.



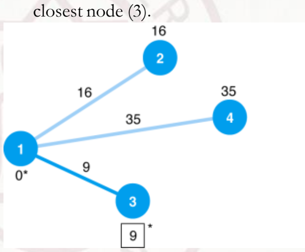
The Shortest Route Problem
Definition and Example Problem Data (2 of 2)

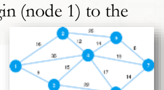


Source: Garn (2010), Issues in Operations Management Figure 7.3 Network Representation

The Shortest Route Problem
Solution Approach (1 of 8)

Determine the initial shortest route from the origin (node 1) to the closest node (3).



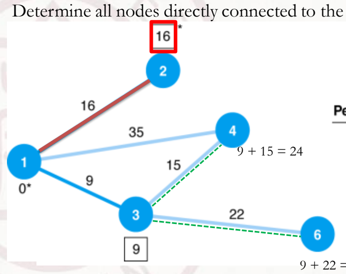


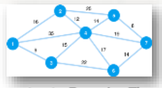
Permanent set	Branch	Time
{1}	1–2	16
	1–4	35
	1–3	$\boxed{9}$*

Figure 7.4 Network with Node 1 in the Permanent Set

The Shortest Route Problem
Solution Approach (2 of 8)

Determine all nodes directly connected to the permanent set.





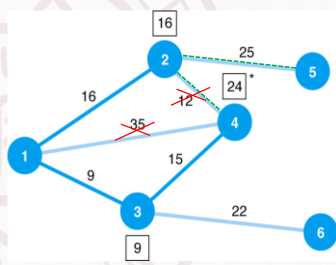
Permanent set	Branch	Time
{1, 3}	1–2	$\boxed{16}$*
	1–4	35
	3–4	24
	3–6	31

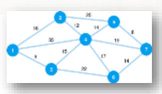
Note that Arc (1, 3) is not anymore in the set of branches.

Figure 7.5 Network with Nodes 1 and 3 in the Permanent Set

The Shortest Route Problem
Solution Approach (3 of 8)

Redefine the permanent set.

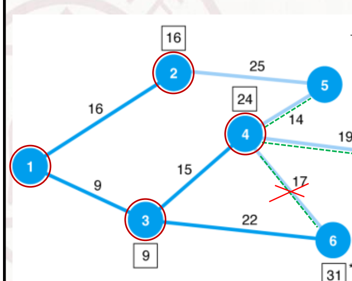




Permanent set	Branch	Time
{1, 2, 3}	~~1-4~~	35
	~~2-4~~	28
	2–5	41
	3–4	$\boxed{24}$*
	3–6	31

Figure 7.6 Network with Nodes 1, 2, and 3 in the Permanent Set

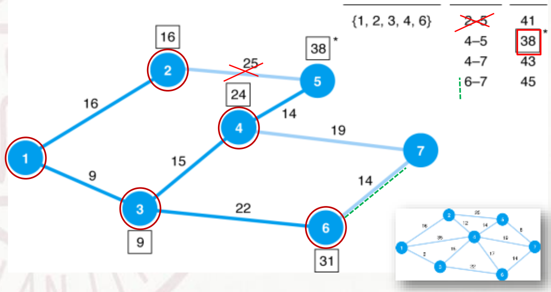
The Shortest Route Problem
Solution Approach (4 of 8)



Permanent set	Branch	Time
{1, 2, 3, 4}	2–5	41
	3–6	$\boxed{31}$*
	4–5	38
	4–7	43
	~~4-6~~	41

Figure 7.7 Network with Nodes 1, 2, 3, and 4 in the Permanent Set

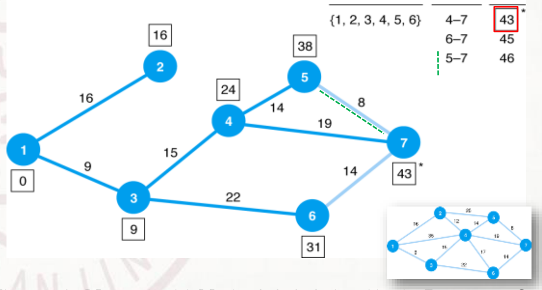
The Shortest Route Problem
Solution Approach (5 of 8)



Permanent set	Branch	Time
{1, 2, 3, 4, 6}	~~2–5~~	41
	4–5	$\boxed{38}$*
	4–7	43
	6–7	45

Figure 7.8 Network with Nodes 1, 2, 3, 4, & 6 in the Permanent Set

The Shortest Route Problem
Solution Approach (6 of 8)



Permanent set	Branch	Time
{1, 2, 3, 4, 5, 6}	4–7	$\boxed{43}$*
	6–7	45
	5–7	46

Figure 7.9 Network with Nodes 1, 2, 3, 4, 5 & 6 in the Permanent Set

The Shortest Route Problem
Solution Approach (7 of 8)

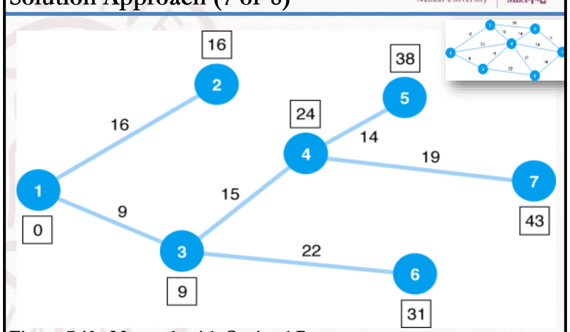


Figure 7.10 Network with Optimal Routes

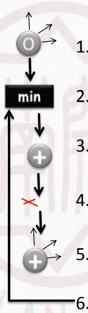
The Shortest Route Problem Solution Approach (8 of 8)

From Los Angeles to:	Route	Total Hours
Salt Lake City (node 2)	1–2	16
Phoenix (node 3)	1–3	9
Denver (node 4)	1–3–4	24
Des Moines (node 5)	1–3–4–5	38
Dallas (node 6)	1–3–6	31
St. Louis (node 7)	1–3–4–7	43

Table 7.1 Shortest Travel Time from Origin to Each Destination

Winter 2016 ML&SCM 13

Shortest Route



1. Put origin in permanent node set and all its outgoing arcs plus "distances" into the branch "analysis" list
2. Look for the minimal distance in this list, this identifies the new arc and node
3. Add the new node to the permanent set
4. Tidy up, i.e. remove arcs that go to the new node
5. Add all its outgoing arcs plus "distances" into the branch "analysis" list
6. Goto step 2 – until all nodes are in the permanent set

Winter 2016 ML&SCM 14

Dijkstra

Algorithm 1 Dijkstra.

Require: network $G = (N, A)$; start node s; sparse cost matrix $C = (c_{ij})$
Ensure: distances d, predecessors p

$S := \emptyset$; $\bar{S} := N$
$d(i) := \infty$ for each node $i \in N$;
$d(s) := 0$; $p(s) := 0$;
while $|S| \leq n$ **do**
  let $i \in \bar{S}$ be a node for which $d(i) = \min\{d(j) : j \in \bar{S}\}$;
  $S := S \cup \{i\}$;
  $\bar{S} := \bar{S} - \{i\}$;
  for each $(i, j) \in A(i)$ **do**
    if $d(j) > d(i) + c_{ij}$ **then**
      $d(j) := d(i) + c_{ij}$; p(j):=i;
    end if
  end for
end while

Winter 2016 ML&SCM 15

The Shortest Route Problem Integer program

Formulation as a 0 - 1 integer program (for one destination only)

$x_{ij} = 0$ if branch i-j is not selected as part of the shortest route and 1 if it is selected.

Minimize $Z = 16x_{12} + 9x_{13} + 35x_{14} + 12x_{24} + 25x_{25} + 15x_{34} + 22x_{36} + 14x_{45} + 17x_{46} + 19x_{47} + 8x_{57} + 14x_{67}$

subject to:

$x_{12} + x_{13} + x_{14} = 1$ — Flow out of source

$x_{12} - x_{24} - x_{25} = 0$
$x_{13} - x_{34} - x_{36} = 0$
$x_{14} + x_{24} + x_{34} - x_{45} - x_{46} - x_{47} = 0$
$x_{25} + x_{45} - x_{57} = 0$
$x_{36} + x_{46} - x_{67} = 0$

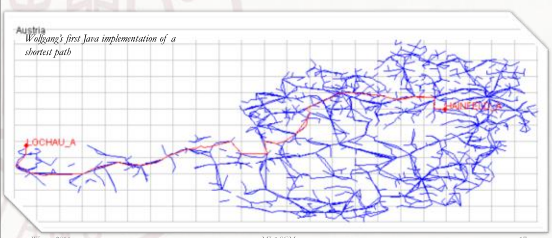
Conservation-of-flow (Note: directed graph)

$x_{47} + x_{57} + x_{67} = 1$ — Flow into destination

$x_{ij} = 0$ or 1

Application *k*-Shortest Path

- Telecommunication – leased lines
- Banks need dedicated line between branches



Bees' tiny brains beat computers – guardian.co.uk

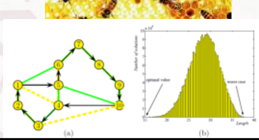
- "Bees can solve complex mathematical problems which keep computers busy for days, research has shown" – 24th October 2010

Dr Nigel Raine, Royal Holloway, University of London

To be pulished in: *The American Naturalist*

How many flowers were used?
- 10 => 10! = 10x9x8x...x1 = 3,628,800 possible solutions

How many bees helped solving the problem?
- Multi Agent Systems, e.g. "Ant algorithms"



Travelling Salesman Problem (TSP)

More applications


- SatNav
  - Most popular application
- Conveyor Systems
  - Collect all products for inventory
- Sorting Centres (Royal Mail)
  - What's the shortest path in centre?
- Warehouses
  - How to pick goods for a delivery? (TSP)

LANHAM ASSOCIATES Supply Chain Excellence

Winter 2016 ML&SCM 19

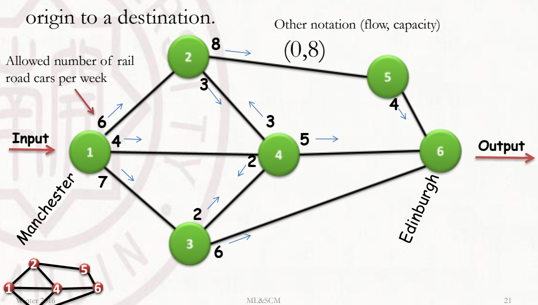
Contents

- ❖ Shortest path
  - ➢ An example
  - ➢ Algorithm
  - ➢ Integer Program
  - ➢ Applications
- ❖ Maximal Flow
  - ➢ An example
  - ➢ Algorithm
  - ➢ Integer Program
  - ➢ Applications


Winter 2016 ML&SCM 20

The Maximal Flow Problem
Definition and Example Problem Data

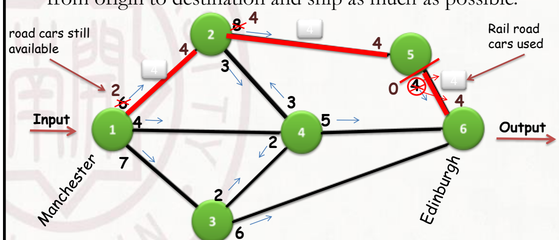
Problem: Maximize the amount of flow of items from an origin to a destination.



Winter 2016 ML&SCM 21

The Maximal Flow Problem
Solution Approach (1 of 5)

Step 1: Arbitrarily choose any path through the network from origin to destination and ship as much as possible.



Maximal Flow for Path 1-2-5-6

The Maximal Flow Problem
Solution Approach (2 of 5)

Step 2: Re-compute branch flow in both directions

Step 3: Select **other feasible paths** arbitrarily and determine maximum flow along the paths until flow is no longer possible.

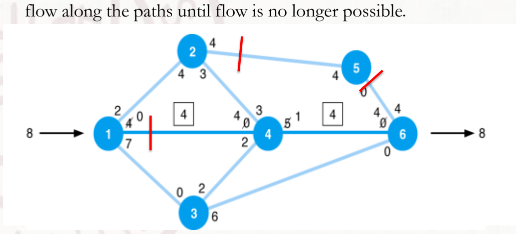


Figure 7.20 Maximal Flow for Path 1-4-6

The Maximal Flow Problem
Solution Approach (3 of 5)

Continue

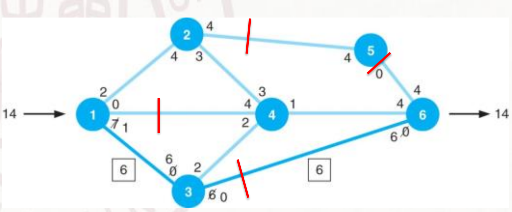


Figure 7.21 Maximal Flow for Path 1-3-6

The Maximal Flow Problem
Solution Approach (4 of 5)

Continue

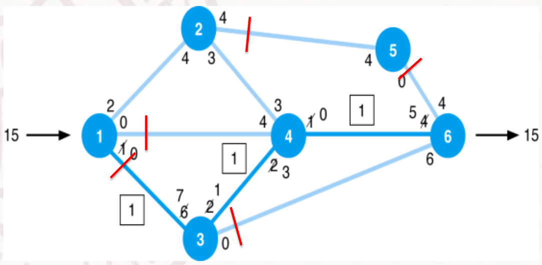


Figure 7.22 Maximal Flow for Path 1-3-4-6

The Maximal Flow Problem
Solution Approach (5 of 5)

Optimal Solution

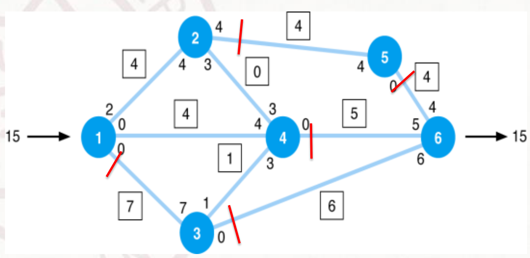


Figure 7.23 Maximal Flow for Railway Network

The Maximal Flow Problem
Solution Method Summary

1. Arbitrarily select any path in the network from origin to destination.
2. Adjust the capacities at each node by subtracting the maximal flow for the path selected in step 1.
3. Add the maximal flow along the path to the flow in the opposite direction at each node.
4. Repeat steps 1, 2, and 3 until there are no more paths with available flow capacity.

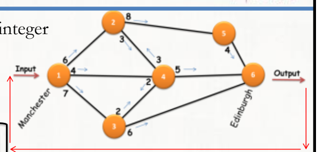
Dantzig and Fulkerson, 1955
"Railnetwork"

The Maximal Flow Problem Integer Program

x_{ij} = flow along branch i-j and integer

Maximize $Z = x_{61}$

subject to:



Flow in & out of source

$x_{61} - x_{12} - x_{13} - x_{14} = 0$
$x_{12} - x_{24} - x_{25} = 0$
$x_{13} - x_{34} - x_{36} = 0$
$x_{14} + x_{24} + x_{34} - x_{46} = 0$
$x_{25} - x_{56} = 0$
$x_{36} + x_{46} + x_{56} - x_{61} = 0$ Flow in & out of destination

Flow out of source is the same as the flow into the destination

Conservation of flow (also known as mass balance constraints)

$x_{12} \le 6$ $x_{24} \le 3$ $x_{34} \le 2$
$x_{13} \le 7$ $x_{25} \le 8$ $x_{36} \le 6$
$x_{14} \le 4$ $x_{46} \le 5$ $x_{56} \le 4$
$x_{61} \le 17$ $x_{ij} \ge 0$ and integer

Note: some opposite flows are missing (for simplification reasons only)

General formulation

maximise v subject to

$$\sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = \begin{cases} v & \text{for } i = s \\ 0 & \text{for all } i \in N - \{s \text{ and } t\} \\ -v & \text{for } i = t \end{cases}$$

$0 \le x_{ij} \le u_{ij}$ for each $(i,j) \in A$

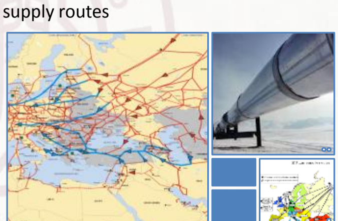
$x = \{x_{ij}\}$ flow, v value of flow

Assumptions:
- The network is directed
- All capacities are nonnegative integers
- There is a path between s and t with finite capacity

More assumptions:
- No directed path from s to t entirely with infinite capacities
- If $(i,j) \in A$ then $(j,i) \in A$
- No parallel arcs

Applications

- Flow of water, electricity, gas and oil
  - E.g. provide required gas to London using all available supply routes



EU Gas network Pipeline Strategic view

Applications

- Traffic flow, Railway system
  - Transport as many goods as possible with time being constraint
  - Joint car navigation system
- Flow of Products through a production line, conveyor systems or sorting centre
  - Maximise letters sorted using all available paths



Winter 2016 ML&SCM 31

Applications

- Tanker Scheduling (Dantzig and Fulkerson, 1954)
- Scheduling of parallel Machines (Federgruen, 1986)
- Distributed Computing (Stone, 1977)


Winter 2016 ML&SCM 32

Recap

- What is the shortest path?
  - A path from a source node to a destination node that minimises the flow.
- What is the maximal flow?
  - A network utilisation that maximises the flow from a source node to a destination node.
- What is probably the most popular shortest path commercial product?
  - SatNav

Winter 2016 ML&SCM 33

The End

- Any questions?

Winter 2016 ML&SCM 34

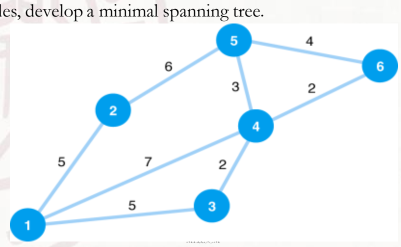
Appendix

- Another shortest route example
- Shortest route in Excel
- Maximal flow in Excel

Winter 2016 ML&SCM 35

Problem Statement and Data

1. Determine the shortest route from Atlanta (node 1) to each of the other five nodes (branches show travel time between nodes).
2. Assume branches show distance (instead of travel time) between nodes, develop a minimal spanning tree.



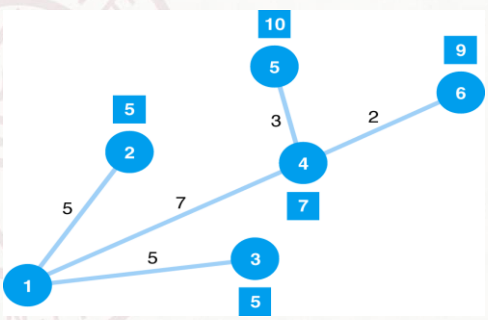
Winter 2016 36

Shortest Route Solution (1 of 2)

Step 1 (part A): Determine the Shortest Route Solution

	Permanent Set	Branch	Time
1.	{1}	1-2	[5]
		1-3	5
		1-4	7
2.	{1,2}	1-3	[5]
		1-4	7
		2-5	11
3.	{1,2,3}	1-4	[7]
		2-5	11
		3-4	7
4.	{1,2,3,4}	4-5	10
		4-6	[9]
5.	{1,2,3,4,6}	4-5	[10]
		6-5	13
6.	{1,2,3,4,5,6}		

Route Solution (2 of 2)



The Shortest Route Problem Integer program

Formulation as a 0 - 1 integer linear programming problem.

$x_{ij} = 0$ if branch i-j is not selected as part of the shortest route and 1 if it is selected.

Minimize $Z = 16x_{12} + 9x_{13} + 35x_{14} + 12x_{24} + 25x_{25} + 15x_{34} + 22x_{36} + 14x_{45} + 17x_{46} + 19x_{47} + 8x_{57} + 14x_{67}$

subject to:

$x_{12} + x_{13} + x_{14} = 1$ → Flow out of source

$$\left.\begin{aligned} x_{12} - x_{24} - x_{25} &= 0 \\ x_{13} - x_{34} - x_{36} &= 0 \\ x_{14} + x_{24} + x_{34} - x_{45} - x_{46} - x_{47} &= 0 \\ x_{25} + x_{45} - x_{57} &= 0 \\ x_{36} + x_{46} - x_{67} &= 0 \end{aligned}\right\} \text{Conservation-of-flow}$$

$x_{47} + x_{57} + x_{67} = 1$ → Flow into destination

$x_{ij} = 0$ or 1

The Shortest Route Problem
Computer Solution with Excel (2 of 4)

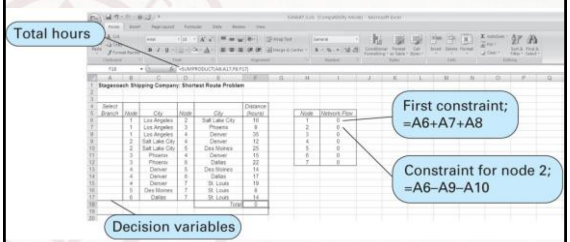


Exhibit 7.3

The Shortest Route Problem
Computer Solution with Excel (3 of 4)

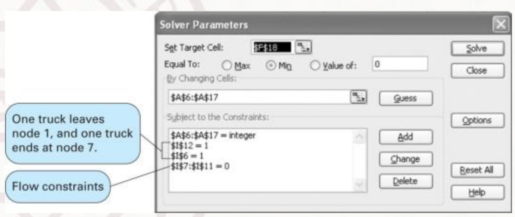


Exhibit 7.4

The Shortest Route Problem
Computer Solution with Excel (4 of 4)

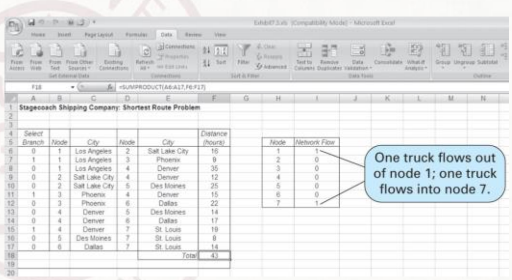


Exhibit 7.5

The Maximal Flow Problem
Computer Solution with Excel (1 of 4)

x_{ij} = flow along branch i-j and integer

Maximize $Z = x_{61}$

subject to:

$x_{61} - x_{12} - x_{13} - x_{14} = 0$
$x_{12} - x_{24} - x_{25} = 0$
$x_{13} - x_{34} - x_{36} = 0$
$x_{14} + x_{24} + x_{34} - x_{46} = 0$
$x_{25} - x_{56} = 0$
$x_{36} + x_{46} + x_{56} - x_{61} = 0$
$x_{12} \leq 6 \quad x_{24} \leq 3 \quad x_{34} \leq 2$
$x_{13} \leq 7 \quad x_{25} \leq 8 \quad x_{36} \leq 6$
$x_{14} \leq 4 \quad x_{46} \leq 5 \quad x_{56} \leq 4$
$x_{61} \leq 17 \quad x_{ij} \geq 0$ and integer

The Maximal Flow Problem
Computer Solution with Excel (2 of 4)

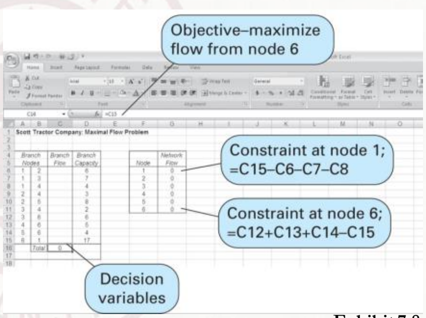


Exhibit 7.8

The Maximal Flow Problem
Computer Solution with Excel (3 of 4)

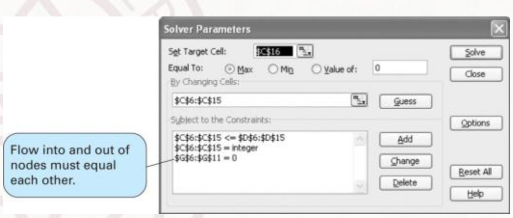
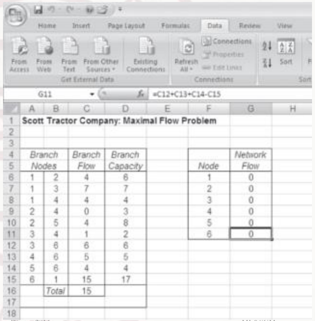


Exhibit 7.9

The Maximal Flow Problem
Computer Solution with Excel (4 of 4)



Branch Nodes		Branch Flow	Branch Capacity
1	2	4	6
1	3	7	7
1	4	4	4
2	4	0	3
2	5	4	8
3	4	1	2
3	6	6	6
4	6	5	5
5	6	4	4
6	1	15	17
	Total	15	

Node	Network Flow
1	0
2	0
3	0
4	0
5	0
6	0

Exhibit 7.10

The Shortest Route Problem
Solution Method Summary

1. Select the node with the shortest direct route from the origin.
2. Establish a permanent set with the origin node and the node that was selected in step 1.
3. Determine all nodes directly connected to the permanent set of nodes.
4. Select the node with the shortest route from the group of nodes directly connected to the permanent set of nodes.
5. Repeat steps 3 & 4 until all nodes have joined the permanent set.

Shortest Route - Algorithm

Dijkstra, 1959

INPUT: *nodes N, Arcs A, Costs C*

OUTPUT: *shortest paths (predecessor notation)*

$S := \emptyset, \overline{S} := N$

$\forall i \in N : p_i := 0, d_i := \infty$ set predecessors and node distances

$d_s := 0$ distance from source

while $|S| < n$ until all paths are found

$i \in \overline{S} : d_i := \min_{j \in \overline{S}} d_j$ find node with minimal distance

$S := S \cup i, \overline{S} := \overline{S} \setminus i$ swap element between sets

$\forall (i,j) \in A_i :$ check all arcs going out from i

$d_j > d_i + c_{ij} \Rightarrow$ if distance can be reduced

$d_j := d_i + c_{ij}, p_j := p_i$ then set distance and predecessor