Modern Logistics & Supply Chain Management

ML & SCM

Production Mix
Mathematical Programming

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Winter 2016

As gold which he cannot spend
will make no man rich,
so knowledge which he cannot apply
will make no man wise.
Samuel Johnson: The Idler No. 84

Learning Objectives

- To decide about production strategies
- To formulate mathematical programs
- To graphically solve linear programs
- Being able to identify mathematical programming problems

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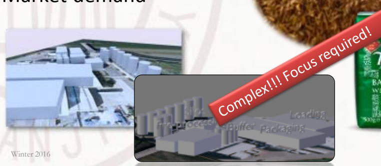
Business Environment

- Silos with raw material
- Factory layout & operations
- Machines and employees
- Finished goods
- Market demand



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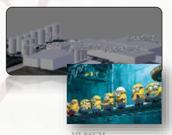
Problem scope

- What is the best production strategy?
  - How many pallets of pure and wild Basmati rice should we produce?
  - So that we maximise the profit!
- What is the production time/capacity?
- How much raw material is available?






Information "gathering"

- Time, material and money
  - All per unit (=pallet in this example)

	Product	production time	raw material	Profit
	Unit	hours per pallet	lb per pallet	£ per pallet
x_1	Pure Basmati rice	1 h/p	4 lb/p	£40/p
x_2	Wild Basmati rice	2 h/p	3 lb/p	£50/p

Mathematical model formulation

- **Step 1** : Clearly define the decision variables
- **Step 2** : Construct the objective function
- **Step 3** : Formulate the constraints

LP Model Formulation A Maximization Example (3 of 4)

Resource Availability
- 40 hour of time per day
- 120 pounds of raw material per day

Decision Variables
- x_1 = number of pure basmati rice pallets to be produced per day
- x_2 = number of wild basmati rice pallets to be produced per day

Objective Function
- Maximise $Z = £40x_1 + £50x_2$
- Z is the profit per day

Resource Constraints
- $1x_1 + 2x_2 \leq 40$ hours … time limit per day
- $4x_1 + 3x_2 \leq 120$ pounds of rice

Non-Negativity Constraints
- $x_1 \geq 0$; $x_2 \geq 0$

Note: This is actually an IP problem and the LP formulation is only "appropriate" if partially filled pallets are feasible.

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The Linear Program

- In a "nice" concise form

maximise

$$Z = £40x_1 + £50x_2$$

subject to:

$$1x_1 + 2x_2 \leq 40$$
$$4x_1 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$

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A feasible solution

- Based on collected data

Product	production time	raw material	Profit
Unit	hours per pallet	lbs per pallet	£ per pallet
Pure Basmati rice	1 h/p	4 lb/p	£40/p
Wild Basmati rice	2 h/p	3 lb/p	£50/p

- A solution

The solution

solution value

Product	Time	raw material	Profit		Volume
Unit	hours	lbs	£		pallet
Pure Basmati rice	5 h	20 lb	£	200	**5 p**
Wild Basmati rice	20 h	30 lb	£	500	**10 p**
consumed	*25 h*	*50 lb*	***£***	***700***	*15 p*
Limit	*40 h*	*120 lb*			
Slack	*15 h*	*70 lb*			

reasons for feasibility

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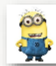
Formal checks

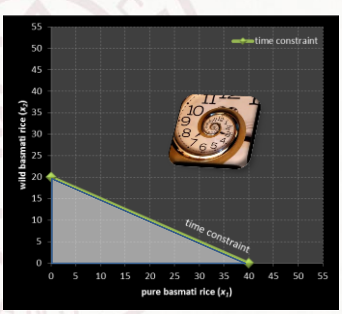
- A feasible solution does not violate any constraints
  - Production time constraint
    - $1x_1 + 2x_2 \le 40$
    - $1 \cdot 5 + 2 \cdot 10 = 25 \le 40$
  - Material constraint
    - $4x_1 + 3x_2 \le 120$
    - $4 \cdot 5 + 3 \cdot 10 = 50 \le 120$
  - Non-Negativity constraints
    - $x_1 = 5 \ge 0;\ \ x_2 = 10 \ge 0$



An Infeasible solution

- At least one constraint is violated
- Let the “solution” be
  - $x_1 = 20$
  - $x_2 = 30$
- Production time constraint
  - $1x_1 + 2x_2 \le 40$
  - $1 \cdot 20 + 2 \cdot 30 = 80 \le 40$

Time constraint

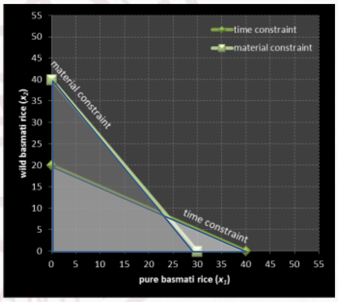


Line shows solutions that are on the time limit

Area under line includes solutions within time limit

$1x_1 + 2x_2 \le 40$
$x_1, x_2 \ge 0$

Time & material constraint

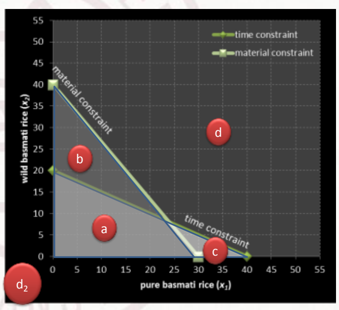


Material constraint

$4x_1 + 3x_2 \leq 120$

An additional limit on the amounts of pallets that can be produced

Game of feasibility



Which region is feasible?

a

Which regions are infeasible?

b, c, d, d_2

An Infeasible Problem

Every possible solution **violates** at least one constraint

Maximize $Z = 5x_1 + 3x_2$

subject to: $4x_1 + 2x_2 \leq 8$

$x_1 \geq 4$

$x_2 \geq 6$

$x_1, x_2 \geq 0$

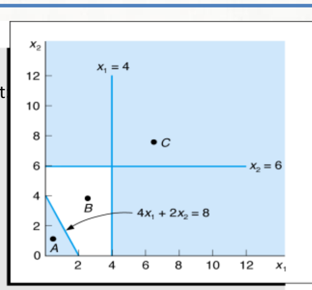


Figure 2.21 Graph of an Infeasible Problem

An Unbounded Problem

Value of the objective function increases indefinitely:

Maximize $Z = 4x_1 + 2x_2$
subject to: $x_1 \geq 4$
$x_2 \leq 2$
$x_1, x_2 \geq 0$

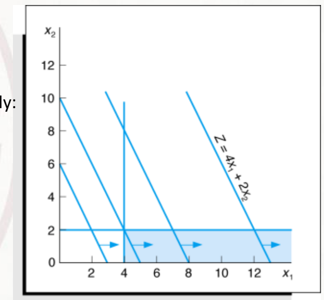


Figure 2.22 Graph of an Unbounded Problem

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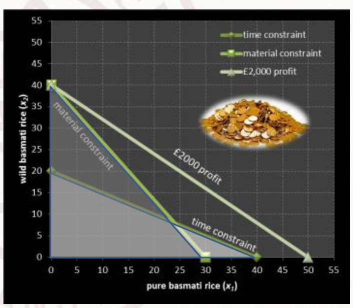
£2,000 Objective



Objective function for a specific value is a line

$Z = £40x_1 + £50x_2$
$= £2{,}000$

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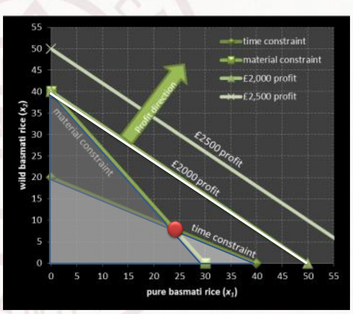
Direction of objective function



Nature of objective function

Where is the feasible solution, that maximises the profit?

In a corner

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The solution

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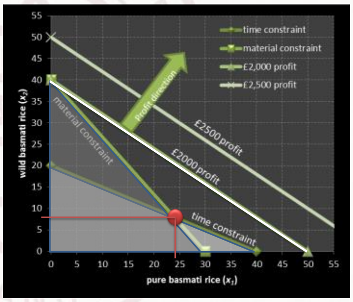

Intersection of profit line with feasible region!

$x_1 = 24$, $x_2 = 8$ } solution

Solution value:

$Z = £40x_1 + £50x_2 =$
$£40 \cdot 24 + £50 \cdot 8 =$
<u>£1,360</u>

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Implementation

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- Production strategy
  - Produce 24 pallets of **pure** Basmati rice per day

  - Produce 8 pallets of **wild** Basmati rice per day

- Maximise profit: £1,360 per day

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Matrix form

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maximise

$40x_1 + 50x_2$ ⇨ $\begin{pmatrix} 40 & 50 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

subject to:

$1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$ ⇨ $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 40 \\ 120 \end{pmatrix}$

$x_1, x_2 \geq 0$ ⇨ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

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Implementation in R

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```
# Production Mix
library(lpSolve) #load the LP library

objective = c(40, 50) #build objective function

#constraints
lhs <- matrix( c(1,2,  4,3), ncol=2, byrow = TRUE)
dir <- c("<=","<=")
rhs <- c(40, 120)

# solve the LP
solvedLP = lp("max", objective, lhs,dir,rhs)

# display the solution and its value
solvedLP$solution
solvedLP$objval
```

$$(40 \quad 50)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 40 \\ 120 \end{pmatrix}$$

Most popular open-source MILP solvers
- lp_solve
- CLP
- GLPK

[CRAN Task View: Optimization and Mathematical Programming]
Contains a list of packages for solving optimisation problems

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Matrix form – more abstract

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- maximise

$$(40 \quad 50)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow cx = \sum_{j=1}^{2} c_j x_j$$

- subject to:

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 40 \\ 120 \\ 0 \\ 0 \end{pmatrix} \Rightarrow Ax \leq b$$

$$\sum_{j=1}^{2} a_{ij} x_j \leq b_i \quad \forall i$$

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Mathematical programs

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Linear Program

$$max\{cx \ : Ax \leq \mathrm{b}, x \in \mathbb{R}^{n\times 1}\}$$
$$A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^{m\times 1}, c \in \mathbb{R}^{1\times n}$$

Integer Program

$$max\{dy \ : By \leq \mathrm{b}, y \in \mathbb{Z}^{k\times 1}\}$$
$$B \in \mathbb{R}^{m\times k}, b \in \mathbb{R}^{m\times 1}, d \in \mathbb{R}^{1\times k}$$

Mixed Integer Program

$$max\{cx + dy \ : Ax + By \leq \mathrm{b}, x \in \mathbb{R}^{m\times 1}, y \in \mathbb{Z}^{k\times 1}\}$$
$$A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{m\times k}, b \in \mathbb{R}^{m\times 1}, c \in \mathbb{R}^{1\times n}, d \in \mathbb{R}^{1\times k}$$

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Mathematical programs

Mixed Integer Program

$$\max_{A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{m\times k}, b \in \mathbb{R}^{m\times 1}, c \in \mathbb{R}^{1\times n}, d \in \mathbb{R}^{1\times k}} \{cx + dy \;:\; Ax + By \le \mathrm{b}, x \in \mathbb{R}^{m\times 1}, y \in \mathbb{Z}^{k\times 1}\}$$

Feasible region

$$S = \{x \in \mathbb{R}^{m\times 1}, y \in \mathbb{Z}^{k\times 1}, Ax + By \le \mathrm{b}\}$$

Feasible solution

$$(x, y) \in S$$

Optimal solution (x^*, y^*)

$$cx^* + dy^* \ge cx + dy \;\; \forall (x, y) \in S$$

Slack Variables

- Standard form requires that all constraints be in the form of equations (equalities).
- A slack variable is ***added to a ≤ constraint*** (weak inequality) to convert it to an equation (=).
- A slack variable typically represents an ***unused resource***.
- A slack variable ***contributes nothing*** to the objective function value.

Linear Programming Model: Standard Form

Max Z = $40x_1 + 50x_2 + 0s_1 + 0s_2$
subject to:
$1x_1 + 2x_2 + s_1 = 40$
$4x_1 + 3x_2 + s_2 = 120$
$x_1, x_2, s_1, s_2 \ge 0$
where: s_1, s_2 are slack variables

- x_1 = number of pure basmati rice pallets to be produced per day
- x_2 = number of wild basmati rice pallets to be produced per day

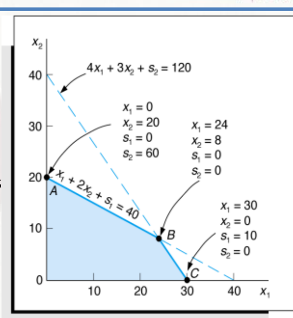
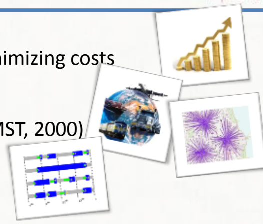


Figure 2.14 Solution Points A, B, and C with Slack

Applications



- Maximizing Profit & minimizing costs
- Transportation Models
- Warehouse locations (MST, 2000)
- Scheduling
- Particular Applications
  - Kellogg's: Production, Inventory and Distribution (2001)
  - UPS Optimizes Its Air Network (2004)
  - Improving Customer Service Operations at Amazon.com (2006)

Recap – "the end is near"

- What helps a mathematical program with?
  – to formulate a problem in a concise, precise and logical manner
- What is mathematical programming?
  – A method to determine an optimal solution
- What can LP, IP or MIP be used for in general?
  – Any linear maximisation or minimisation problem that is subject to linear constraints
- Do you know any concrete applications?
  – Production mix, scheduling, transportation problems, …

The End

- Any questions?

Appendix

- Maximisation Example — Similar to the "rice" example
- Slack Variables — Explains unused resources
- Minimisation — Approach and method the same as maximisation (except objective)
- Possible solutions — Terminology and a more complete view
- Yet another example — More practice is always good

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Model components

- **Decision variables**
  - Examples: what quantity to produce, who should be doing the job, …
  - mathematical symbols representing levels of activity of a firm
- **Objective function**
  - Examples: maximise profit, maximise flow, minimise costs, …
  - a linear mathematical relationship describing an objective of the firm in terms of decision variables
    - this function is to be maximised or minimised.
- **Constraints**
  - Examples: available material is limited, production time limited
  - requirements or restrictions placed on the firm by the operating environment
  - stated in linear relationships of the decision variables
- **Parameters**
  - Examples: price per unit, total available raw material, total hours available
  - numerical coefficients and constants used in the objective function and constraints.

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Mathematical Programming

- How do I solve such a business problem?
  1. Identify problem as solvable by linear programming.
  2. Formulate a mathematical model of the unstructured problem.
  3. Solve the model.
  4. Implementation.

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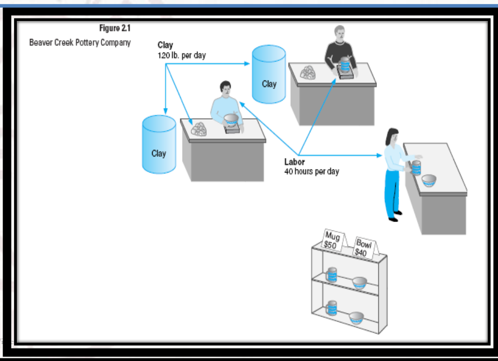
LP Model Formulation
A Maximization Example (1 of 4)



LP Model Formulation
A Maximization Example (2 of 4)

- Product mix problem - Beaver Creek Pottery Company
- How many bowls and mugs should be produced to maximize profits given labor and materials constraints?
- Product resource requirements and unit profit:

	Resource Requirements		
Product	Labor (Hr./Unit)	Clay (Lb./Unit)	Profit (£/Unit)
Bowl	1	4	40
Mug	2	3	50

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LP Model Formulation
A Maximization Example (3 of 4)

Resource Availability: 40 hrs of labor per day
120 lbs of clay

Decision Variables: x_1 = number of bowls to produce per day
x_2 = number of mugs to produce per day

Objective Function: Maximize $Z = £40x_1 + £50x_2$
Where Z = profit per day

Resource Constraints: $1x_1 + 2x_2 \leq 40$ hours of labor
$4x_1 + 3x_2 \leq 120$ pounds of clay

Non-Negativity Constraints: $x_1 \geq 0$; $x_2 \geq 0$

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LP Model Formulation
A Maximization Example (4 of 4)

Complete Linear Programming Model:

Maximize $Z = £40x_1 + £50x_2$

subject to: $1x_1 + 2x_2 \leq 40$
$4x_2 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

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Feasible Solutions

A ***feasible solution*** does not violate ***any*** of the constraints:

Example: $x_1 = 5$ bowls
$x_2 = 10$ mugs
$Z = £40x_1 + £50x_2 = £700$

Labor constraint check: 1(5) + 2(10) = 25 < 40 hours
Clay constraint check: 4(5) + 3(10) = 70 < 120 pounds

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Infeasible Solutions

An ***infeasible solution*** violates ***at least one*** of the constraints:

Example: $x_1 = 10$ bowls
$x_2 = 20$ mugs
$Z = £40x_1 + £50x_2 = £1400$

Labor constraint check: 1(10) + 2(20) = 50 > 40 hours

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Coordinate Axes
Graphical Solution of Maximization Model (1 of 12)

X2 is mugs

Maximize Z = £40x_1 + £50x_2
subject to: $1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

X1 is bowls

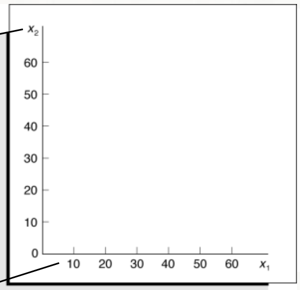

Figure 2.2 Coordinates for Graphical Analysis

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Labor Constraint
Graphical Solution of Maximization Model (2 of 12)

Maximize Z = £40x_1 + £50x_2
subject to: $1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

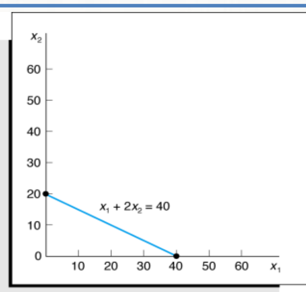

Figure 2.3 Graph of Labor Constraint

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Labor Constraint Area
Graphical Solution of Maximization Model (3 of 12)

Maximize Z = £40x_1 + £50x_2
subject to: $1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

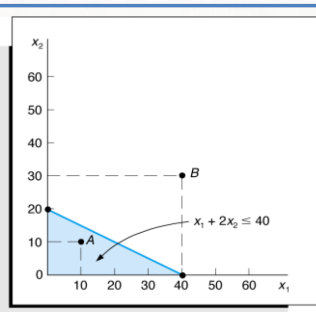

Figure 2.4 Labor Constraint Area

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Clay Constraint Area
Graphical Solution of Maximization Model (4 of 12)

Maximize $Z = £40x_1 + £50x_2$
subject to: $1x_1 + 2x_2 \le 40$
$4x_1 + 3x_2 \le 120$
$x_1, x_2 \ge 0$

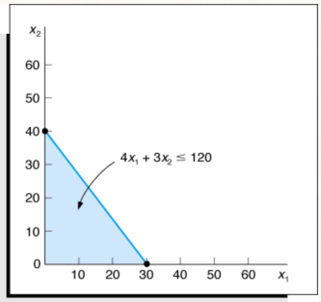

Figure 2.5 Clay Constraint Area

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Both Constraints
Graphical Solution of Maximization Model (5 of 12)

Maximize $Z = £40x_1 + £50x_2$
subject to: $1x_1 + 2x_2 \le 40$
$4x_1 + 3x_2 \le 120$
$x_1, x_2 \ge 0$

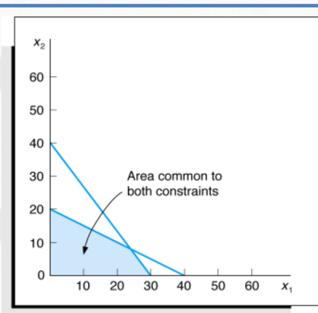

Figure 2.6 Graph of Both Model Constraints

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Feasible Solution Area
Graphical Solution of Maximization Model (6 of 12)

Maximize $Z = £40x_1 + £50x_2$
subject to: $1x_1 + 2x_2 \le 40$
$4x_1 + 3x_2 \le 120$
$x_1, x_2 \ge 0$

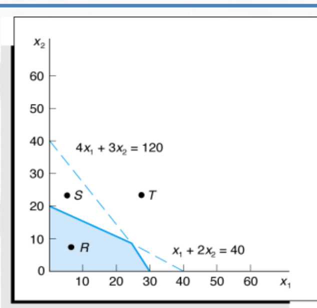

Figure 2.7 Feasible Solution Area

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Objective Function Solution = £800
Graphical Solution of Maximization Model (7 of 12)

Maximize $Z = £40x_1 + £50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

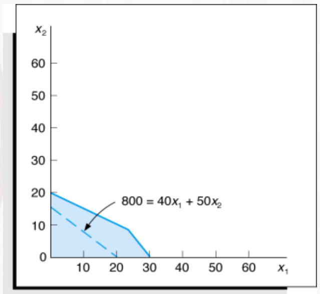

Figure 2.8 Objective Function Line for Z = £800

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Alternative Objective Function Solution Lines
Graphical Solution of Maximization Model (8 of 12)

Maximize $Z = £40x_1 + £50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

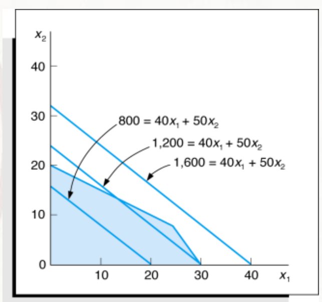

Figure 2.9 Alternative Objective Function Lines

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Optimal Solution
Graphical Solution of Maximization Model (9 of 12)

Maximize $Z = £40x_1 + £50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

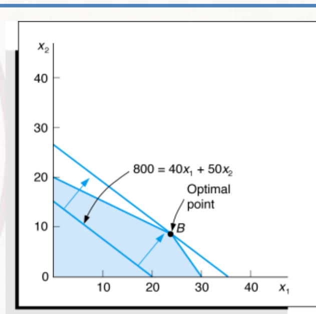

Figure 2.10 Identification of Optimal Solution Point

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Optimal Solution Coordinates
Graphical Solution of Maximization Model (10 of 12)

Maximize Z = £$40x_1$ + £$50x_2$
subject to: $1x_1 + 2x_2 \le 40$
$4x_1 + 3x_2 \le 120$
$x_1, x_2 \ge 0$

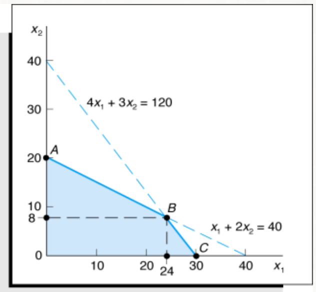

Figure 2.11 Optimal Solution Coordinates

Extreme (Corner) Point Solutions
Graphical Solution of Maximization Model (11 of 12)

Maximize Z = £$40x_1$ + £$50x_2$
subject to: $1x_1 + 2x_2 \le 40$
$4x_1 + 3x_2 \le 120$
$x_1, x_2 \ge 0$

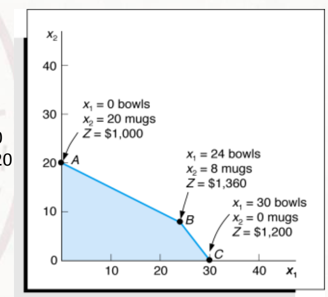

Figure 2.12 Solutions at All Corner Points

Optimal Solution for New Objective Function
Graphical Solution of Maximization Model (12 of 12)

Maximize Z = £$70x_1$ + £$20x_2$
subject to: $1x_1 + 2x_2 \le 40$
$4x_1 + 3x_2 \le 120$
$x_1, x_2 \ge 0$

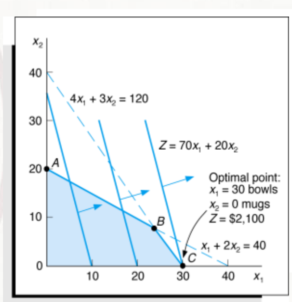

Figure 2.13 Optimal Solution with Z = $70x_1 + 20x_2$

Graphical Solution of LP Models

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- Graphical solution is limited to linear programming models containing ***only two decision variables*** (can be used with three variables but only with great difficulty).
- Graphical methods provide ***visualization of how*** a solution for a linear programming problem is obtained.

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LP Model Formulation – Minimization (1 of 8)

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- Two brands of fertilizer available - Super-gro, Crop-quick.
- Field requires at least 16 pounds of nitrogen and 24 pounds of phosphate.
- Super-gro costs £6 per bag, Crop-quick £3 per bag.
- Problem: How much of each brand to purchase to minimize total cost of fertilizer given following data ?

	Chemical Contribution	
Brand	Nitrogen (lb/bag)	Phosphate (lb/bag)
Super-gro	2	4
Crop-quick	4	3

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LP Model Formulation – Minimization (2 of 8)

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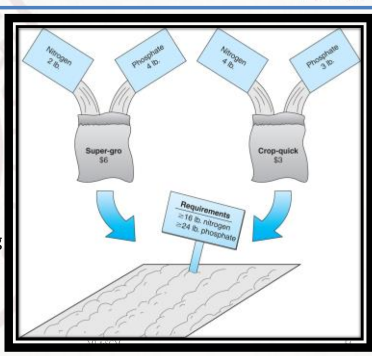

Figure 2.15 Fertilizing farmer's field

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LP Model Formulation – Minimization (3 of 8)

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Decision Variables:

x_1 = bags of Super-gro
x_2 = bags of Crop-quick

The Objective Function:

Minimize $Z = £6x_1 + £3x_2$
Where: $£6x_1$ = cost of bags of Super-Gro
$£3x_2$ = cost of bags of Crop-Quick

Model Constraints:

$2x_1 + 4x_2 \geq 16$ lb (nitrogen constraint)
$4x_1 + 3x_2 \geq 24$ lb (phosphate constraint)
$x_1, x_2 \geq 0$ (non-negativity constraint)

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Constraint Graph – Minimization (4 of 8)

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Minimize $Z = £6x_1 + £3x_2$
subject to: $2x_1 + 4x_2 \geq 16$
$4x_1 + 3x_2 \geq 24$
$x_1, x_2 \geq 0$

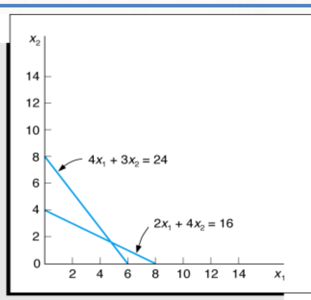

Figure 2.16 Graph of Both Model Constraints

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Feasible Region– Minimization (5 of 8)

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Minimize $Z = £6x_1 + £3x_2$
subject to: $2x_1 + 4x_2 \geq 16$
$4x_1 + 3x_2 \geq 24$
$x_1, x_2 \geq 0$

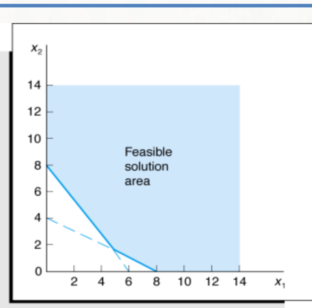

Figure 2.17 Feasible Solution Area

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Optimal Solution Point – Minimization (6 of 8)

Minimize $Z = £6x_1 + £3x_2$
subject to: $2x_1 + 4x_2 \geq 16$
$4x_1 + 3x_2 \geq 24$
$x_1, x_2 \geq 0$

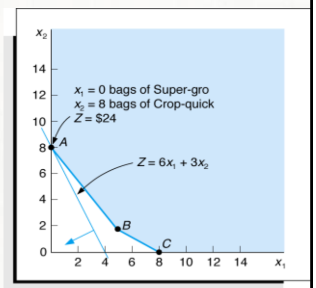

Figure 2.18 Optimum Solution Point

Surplus Variables – Minimization (7 of 8)

- A surplus variable is ***subtracted from a ≥ constraint*** to convert it to an equation (=).
- A surplus variable ***represents an excess*** above a constraint requirement level.
- A surplus variable ***contributes nothing*** to the calculated value of the objective function.
- Subtracting surplus variables in the farmer problem constraints:

$2x_1 + 4x_2 - s_1 = 16$ (nitrogen)
$4x_1 + 3x_2 - s_2 = 24$ (phosphate)

Graphical Solutions – Minimization (8 of 8)

Minimize $Z = £6x_1 + £3x_2 + 0s_1 + 0s_2$
subject to: $2x_1 + 4x_2 - s_1 = 16$
$4x_1 + 3x_2 - s_2 = 24$
$x_1, x_2, s_1, s_2 \geq 0$

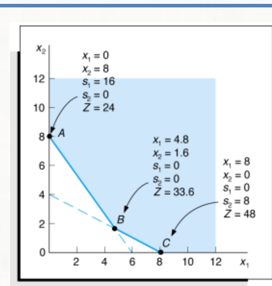

Figure 2.19 Graph of Fertilizer Example

Irregular Types of Linear Programming Problems

For some linear programming models, the general rules do not apply.

- Special types of problems include those with:
  - Multiple optimal solutions
  - Infeasible solutions
  - Unbounded solutions

Multiple Optimal Solutions Beaver Creek Pottery

The objective function is **parallel** to a constraint line.

Maximize $Z = £40x_1 + 30x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Where:

x_1 = number of bowls

x_2 = number of mugs

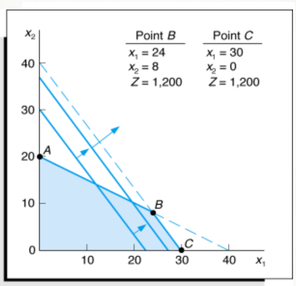

Figure 2.20 Example with Multiple Optimal Solutions

Characteristics of Linear Programming Problems

- A decision amongst alternative courses of action is required.
- The decision is represented in the model by **decision variables**.
- The problem encompasses a goal, expressed as an **objective function**, that the decision maker wants to achieve.
- Restrictions (represented by **constraints)** exist that limit the extent of achievement of the objective.
- The objective and constraints must be definable by **linear** mathematical functional relationships.

Properties of Linear Programming Models

- **Proportionality** - The rate of change (slope) of the objective function and constraint equations is constant.
- **Additivity** - Terms in the objective function and constraint equations must be additive.
- **Divisibility** -Decision variables can take on any fractional value and are therefore continuous as opposed to integer in nature.
- **Certainty** - Values of all the model parameters are assumed to be known with certainty (non-probabilistic).

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Problem Statement
Example Problem No. 1 (1 of 3)

- Hot dog mixture in 1000-pound batches.
- Two ingredients, chicken (£3/lb) and beef (£5/lb).
- Recipe requirements:

  at least 500 pounds of "chicken"

  at least 200 pounds of "beef"
- Ratio of chicken to beef must be at least 2 to 1.
- Determine optimal mixture of ingredients that will minimize costs.

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Solution
Example Problem No. 1 (2 of 3)

Step 1:

Identify decision variables.

x_1 = lb of chicken in mixture

x_2 = lb of beef in mixture

Step 2:

Formulate the objective function.

Minimize $Z = £3x_1 + £5x_2$
where Z = cost per 1,000-lb batch
$£3x_1$ = cost of chicken
$£5x_2$ = cost of beef

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Solution
Example Problem No. 1 (3 of 3)

Step 3:

Establish Model Constraints

$x_1 + x_2 = 1{,}000$ lb

$x_1 \geq 500$ lb of chicken

$x_2 \geq 200$ lb of beef

$x_1/x_2 \geq 2/1$ or $x_1 - 2x_2 \geq 0$

$x_1, x_2 \geq 0$

The Model: Minimize $Z = £3x_1 + 5x_2$

subject to: $x_1 + x_2 = 1{,}000$ lb

$x_1 \geq 50$

$x_2 \geq 200$

$x_1 - 2x_2 \geq 0$

$x_1, x_2 \geq 0$

Example Problem No. 2 (1 of 3)

Solve the following model graphically:

Maximize $Z = 4x_1 + 5x_2$

subject to: $x_1 + 2x_2 \leq 10$

$6x_1 + 6x_2 \leq 36$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Step 1: Plot the constraints as equations

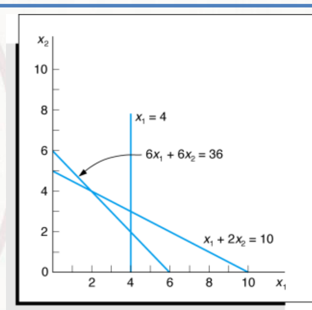

Figure 2.23 Constraint Equations

Example Problem No. 2 (2 of 3)

Maximize $Z = 4x_1 + 5x_2$

subject to: $x_1 + 2x_2 \leq 10$

$6x_1 + 6x_2 \leq 36$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Step 2: Determine the feasible solution space

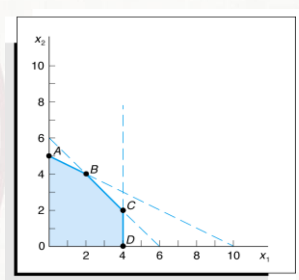

Figure 2.24 Feasible Solution Space and Extreme Points

Example Problem No. 2 (3 of 3)

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Maximize $Z = 4x_1 + 5x_2$
subject to: $x_1 + 2x_2 \leq 10$
$6x_1 + 6x_2 \leq 36$
$x_1 \leq 4$
$x_1, x_2 \geq 0$

Step 3 and 4: Determine the solution points and optimal solution

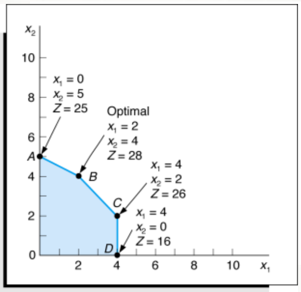

Figure 2.25 Optimal Solution Point

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